AVERAGING OF THE EQUATIONS OF THERMAL-RADIATION TRANSFER IN A PLASMA

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Consideration is given to methods of averaging the equation of radiation transfer for the diffusion and "forward-backward" double-flow approximations. Allowance for the anisotropy of the indicatrices of radiation in the three-temperature approximation is shown to lead to substantial changes in the averaged transfer equations. Actually, the equations involve the temperature difference to the first power, which is significant when problems with nonequilibrium radiation conditions are considered.

In solving a number of gasdynamic problems for which transfer of energy by radiation plays a significant and sometimes a decisive role, the approximation of radiation heat conduction often turns out to be inadequate. This is particularly true of flows in zones of sharp gradients of gasdynamic quantities or in zones with high rates of change of the parameters, typical examples of which are shock waves and rarefaction waves. Bearing in mind the need for working (although modeling in some respects) equations of transfer of energy by radiation as applied to such flows under conditions where a dominant role is played by radiation of the continuous spectrum, we consider the procedure for frequency averaging of spectral equations that describe energy transfer by radiation. First we consider the diffusion approximation and then the double-flow ("forward-backward") approximation.

1. Diffusion Approximation. As is known (for example, [1]) when the anisotropy of the radiation is weak, the radiation transfer equation

$$\frac{\partial U_\nu}{\partial t} + \text{div}\, \overline{S}_\nu = c\kappa_\nu' (U_{\nu\text{eq}} - U_\nu) \tag{1}$$

is supplemented by the equation

$$\overline{S}_\nu = -\frac{c}{3} l_\nu' \nabla U_\nu\,, \quad l_\nu' = \frac{1}{\kappa_\nu'}\,, \tag{2}$$

which is a definition of the spectral radiation flux $\overline{S}_\nu$. In (1) and (2), $U_{\nu\text{eq}}$ and U_ν are the equilibrium and nonequilibrium spectral radiation densities; κ_ν' is the coefficient of absorption at the frequency ν corrected for forced emission; l_ν' is the corresponding radiation mean free path. System (1) and (2) determines completely the radiation in the approximation considered.

The ordinary procedure for averaging the system (1) and (2) over frequencies [1] causes them to be replaced by the equations

$$\frac{\partial U}{\partial t} + \text{div}\, \overline{S} = c\widetilde{\kappa}(U_{\text{eq}} - U)\,, \tag{3}$$

$$\overline{S} = -\frac{c}{3} \widetilde{l} \nabla U\,; \quad \widetilde{l} = 1/\widetilde{\kappa}\,, \tag{4}$$

which, strictly speaking, hold true only in the case of a "gray" substance, i.e., a substance with a frequency-independent absorption coefficient. When the dependence of the absorption coefficient κ_ν on the frequency is strong,

Rosseland or Planck (for small optical thicknesses) averaging of the coefficient κ'_ν and the radiation mean free path l'_ν over the frequency is used; the quantities $\tilde{\kappa}$ and $\tilde{l}$ obtained in this manner are used in Eqs. (3) and (4). Noteworthy is the fact that, in this case, first, the same law of averaging of the factors in the terms $\kappa'_\nu U_{\nu eq}$ and $\kappa'_\nu U_\nu$ in (1) is assumed (the same weighting function) and, second, in (2), l'_ν is averaged with the function $U_{\nu eq}$ instead of U_ν; furthermore, in solving specific problems we often assume $\tilde{\kappa} = 1/\tilde{l}$.

The above makes it necessary to dwell in greater detail on the problem of averaging of Eqs. (1) and (2). First of all, we note the following. The presented procedure for frequency averaging is based on the fact that the weighting functions with which the averaging is performed are obtained from the equilibrium radiation density $U_{\nu eq}$, i.e., in averaging U_ν-containing terms it is assumed in essence that there is no frequency nonequilibrium of the spectrum — $U_{\nu eq}$ and U_ν as functions of the frequency are analogous. Simultaneously, the dependence on the temperature T coincident with the substance temperature that is contained in $U_{\nu eq}$ is used automatically in U_ν. If the assumption of the Planck form of U_ν is the only possible one and it can be made in this case in order to perform averaging in general form without determining U_ν by solving system (1) and (2), the assumption of equality of the radiation temperature T_1 and the substance temperature T is absolutely superfluous since when the function U_ν is prescribed in the Planck form we need only introduce the parameter T_1 that has the meaning of the radiation temperature and is not necessarily equal to T. Use of the diffusion approximation in cases where the difference between $U_{\nu eq}$ and U_ν is large requires that this difference be allowed for in averaging not only the function U itself but also the coefficients in the transfer equations. Indeed, for example, for a bremsstrahlung mechanism of absorption and emission, $\kappa_\nu \sim T^{-3.5}$ and $l_\nu \sim T^{3.5}$, i.e., the temperature dependence of the coefficients in the equations is almost the same as the dependence that is introduced by U_ν after averaging.

On this basis we will average Eqs. (1) and (2) with U_ν and $U_{\nu eq}$ prescribed by the corresponding Planck expressions:

$$U_\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{\exp\left(\dfrac{h\nu}{kT_1}\right) - 1} , \quad U_{\nu eq} = \frac{8\pi h\nu^3}{c^3}\frac{1}{\exp\left(\dfrac{h\nu}{kT}\right) - 1} \tag{5}$$

for the specific case of a gas in the reqion of multiple and total ionization with bremsstrahlung absorption and photoabsorption. The coefficient of total absorption κ'_ν with a correction for forced emission is [1]

$$\kappa'_\nu = \kappa_\nu \left(1 - \exp\left(\frac{h\nu}{kT}\right)\right) . \tag{6}$$

Integrating (1) and (2) on the left and the right over the frequency going from 0 to ∞, we have

$$\frac{\partial U}{\partial t} + \text{div } \overline{S} = c\,(\kappa_{eq}U_{eq} - \kappa U) ,$$

$$U_{eq} = \int\limits_0^\infty U_{\nu eq}d\nu = \frac{4\sigma T^4}{c} ; \quad U = \int\limits_0^\infty U_\nu d\nu = \frac{4\sigma T_1^4}{c} ; \tag{7}$$

$$\overline{S} = -\frac{c}{3}\,l\,\nabla U ; \quad S = \int\limits_0^\infty S_\nu d\nu ; \quad \sigma = \frac{2\pi^5 k^4}{15h^3 c^2} , \tag{8}$$

and by definition

$$\kappa_{eq} = \frac{\int\limits_0^\infty \kappa'_\nu U_{\nu eq}d\nu}{U_{eq}} ; \quad \kappa = \frac{\int\limits_0^\infty \kappa'_\nu U_\nu d\nu}{U} ; \quad l = \frac{\int\limits_0^\infty l'_\nu \dfrac{dU_\nu}{dT_1} d\nu}{\dfrac{dU}{dT_1}} . \tag{9}$$

Next we calculate κ_{eq}, κ, and l in the average-charge approximation according to [2]. Introducing additionally the notation $h\nu/kT_1 = \nu$, $h\nu/kT = u$, $\bar{u}_* = \bar{J}/kT$, and $\beta = T/T_1$, we have

$$\kappa = \kappa_0 \frac{T \int_0^\infty \dfrac{e^{\overline{\Psi}}}{u^3} \dfrac{\nu^3}{\exp\nu - 1}(1 - \exp(-u))\,du}{\dfrac{\pi^4}{15}T_1} = \kappa_0 \frac{15}{\pi^4}\beta^4 \times \tag{10}$$

$$\times \left\{ \int_0^{\bar{u}_*} \exp(u(1-\beta)) \frac{1 - \exp(-u)}{1 - \exp(-\beta u)}\,du + \int_{\bar{u}_*}^\infty \exp(\bar{u}_* - \beta u) \frac{1 - \exp(-u)}{1 - \exp(-\beta u)}\,du \right\},$$

where κ_0 is the absorption coefficient in the average-charge approximation, determined in [2]. Since $\kappa = \kappa_{eq}$ for $\beta = 1$, we find κ_{eq}:

$$\kappa_{eq} = \frac{15}{\pi^4}\kappa_0 (1 + \bar{u}_*). \tag{11}$$

Generally speaking, expression (10) for κ can be simplified in different ways depending on the quantity $\bar{u}_* = \bar{J}/kT$; however, the closeness of β to unity is often used (recall that $\beta = (U_{eq}/U)^{1/4}$). Then, assuming $1 - \exp(-\beta u) = 1 - \exp(-u)$ in the integrand, we obtain

$$\kappa = \frac{15}{\pi^4}\kappa_0 \beta^4 \frac{\exp((1-\beta)u_*) - \beta}{\beta(1-\beta)}. \tag{12}$$

When $\beta \to 1$, (12) becomes (11).

Let us determine l now. From (9) and (6) with the use of the average-charge model we find

$$l = \frac{1}{\kappa_0} \frac{\beta^2 \int_0^\infty \dfrac{u^4 \exp(\beta u)}{(\exp(\beta u) - 1)^2} \dfrac{u^3}{1 - \exp(-u)} \exp(-\overline{\Psi})\,du}{\dfrac{4\pi^4}{15}} =$$

$$= \frac{1}{\kappa_0} \frac{15}{4\pi^4} \beta^2 \int_0^\infty \frac{u^7 \exp(-\overline{\Psi} - \beta u)\,du}{(1 - \exp(-u))(1 - \exp(-\beta u))^2} \approx$$

$$\approx \frac{15}{4\pi^4} \frac{1}{\kappa_0} \beta^2 \left\{ \int_0^{\bar{u}_*} u^7 \exp(-(1+\beta))\,du + \int_{\bar{u}_*}^\infty u^7 \exp(-\bar{u}_* - \beta u)\,du \right\}. \tag{13}$$

Here the final expression is determined by the value of $\bar{u}_*$. We can distinguish three cases: 1) $\bar{u}_* \leq 1$; 2) $3.5 < \bar{u}_* < 7$; 3) $\bar{u}_* > 7$. The result for $\beta \sim 1$ differs little from [2], and we do not give it here.

Let us consider the final expressions for the case $\bar{u}_* \leq 1$ and $\beta \sim 1$. We have

$$\kappa = \frac{15}{\pi^4}\kappa_0\beta^4 \frac{1 + U_*}{\beta} = \kappa_{eq}\beta^3, \quad l \approx \frac{15}{4\pi^4}\frac{1}{\kappa_0}\frac{7!}{\beta^6}\exp(-\bar{u}_*) \approx \frac{30}{\kappa_{eq}}\frac{1}{\beta^6} = l_R^{(T)}\frac{1}{\beta^6}. \tag{14}$$

Then the averaged transfer equations take the form

$$\frac{\partial U}{\partial t} + \mathrm{div}\,\overline{S} = c\kappa_{eq}U_{eq}\left(1 - \frac{1}{\beta}\right), \quad \frac{1}{\beta} = \left(\frac{U}{U_{eq}}\right)^{1/4}, \tag{15}$$

$$\overline{S} = -\frac{c}{3}\frac{30}{\kappa_{\text{eq}}}\left(\frac{U}{U_{\text{eq}}}\right)^6 \nabla U . \tag{16}$$

We compare (15) and (16) with (3) and (4), where, for $\widetilde{\kappa}$ and $\widetilde{l}$, we take the commonly used Rosseland means, which are equal to $\widetilde{\kappa} = 1/\widetilde{l} = 1/l_R^{(T)} = \kappa_{\text{eq}}/30$ for the considered case of photo and bremsstrahlung mechanisms of absorption. It can be seen that when $\beta \approx 1$ Eq. (16) coincides with (4) while (3) differs from (15) both in form and in numerical coefficient — an additional numerical factor of 30 appears in (15) on the right (for $u_* \ll 1$). In the particular case $\beta \to 1$, the right-hand sides of (3) and (15) differ by a factor of ~ 7.5 since $U_{\text{eq}} - U = U_{\text{eq}}(1 - 1/\beta^4)$. We note that the appearance of the indicated factor means an increase in the intensity of energy transfer between the radiation and the substance by the same factor, which can have a substantial effect on the gasdynamics, especially in a region that is nonequilibrium in the radiation. Finally, we note that in the case $\overline{u}_* \leq 1$ and $\beta \to 1$ the indicated difference is smallest. However, conditions where $\overline{J}/kT = \overline{u}_* \gg 1$ are very frequent, the difference increasing strongly. Apart from numerical differences there is, of course, a more significant qualitative difference between systems (15) and (16) and (3) and (4): the radiation density U enters (3) and (4) linearly, i.e., in the quasistationary case ($\partial U/\partial t = 0$) these equations can be integrated formally when needed; in general form system (15) and (16) is not integrated because of the nonlinearity.

The results obtained above indicate that nonequilibrium of the radiation, in a number of cases, can play a significant role and can lead to results that differ quantitatively and qualitatively from the solutions yielded by the transfer equations in the approximation of $T_1 = T$. It is also evident that the averaged transfer equations (3) and (4) can give a distorted representation of the real process of heat exchange between the radiation and the substance since $\widetilde{l} \neq 1/\widetilde{\kappa}$. The two-temperature approximation with temperatures of the substance T and the radiation T_1 that are different from one another is used rather frequently in problems of radiation gasdynamics in the presence of large gradients of the parameters, where the radiation does not come into equilibrium with the substance [3]. In this case, however, it seems more consistent to introduce two temperatures for the radiation field because of its anisotropy. In heating a substance by radiation (or cooling due to luminescence) at a fixed point of space, one-direction flows correspond to radiation with significantly different intensities, i.e., the temperatures T_+ and T_-. We consider this problem using the double-flow approximation as an example [1].

2. Double-Flow Approximation ("Forward-Backward" Approximation). The radiation transfer equations in the double-flow approximation for the plane case can be written in the form [1]

$$\frac{1}{c}\frac{\partial J_\nu^+}{\partial t} + \frac{1}{2}\frac{\partial J_\nu^+}{\partial t} = \kappa_\nu' (J_{\nu\text{eq}} - J_\nu^+) , \tag{17}$$

$$\frac{1}{c}\frac{\partial J_\nu^-}{\partial t} + \frac{1}{2}\frac{\partial J_\nu^-}{\partial t} = \kappa_\nu' (J_{\nu\text{eq}} - J_\nu^-) . \tag{18}$$

Equations (17) and (18) are obtained from the exact equation of radiation transfer

$$\frac{1}{c}\frac{\partial J_\nu}{\partial t} + \overline{\Omega} \nabla J_\nu = \kappa_\nu' (J_{\nu\text{eq}} - J_\nu) \tag{19}$$

by averaging it over the angles in the corresponding hemispheres — separately for the positive and negative x directions; in each hemisphere, the radiation intensity J_ν^+ or J_ν^- belonging to it is assumed to be isotropic but $J_\nu^+ \neq J_\nu^-$. In (17) and (18), $J_{\nu\text{eq}}$ is the equilibrium radiation intensity

$$J_{\nu\text{eq}} = \frac{cU_{\nu\text{eq}}}{4\pi} = \frac{2h\nu^3}{c^2}\frac{1}{\exp\frac{h\nu}{kT} - 1} . \tag{20}$$

The density of nonequilibrium radiation U_ν and the radiation flux S_ν of frequency ν are related to J_ν^+ and J_ν^- by the expressions

$$U_\nu = \frac{2\pi}{c}(J_\nu^+ + J_\nu^-), \quad S_\nu = \pi (J_\nu^+ - J_\nu^-), \tag{21}$$

since by definition $U_\nu = \frac{1}{c}\int J_\nu d\Omega$ and $S_\nu = \int J_\nu \cos\theta d\Omega$, where $d\Omega = 2\pi \sin\theta d\theta$.

We average (17) and (18), assuming an equilibrium form of $J_\nu^\pm$ as a function of frequency:

$$J_\nu^+ = \frac{2\pi\nu^3}{c^2}\frac{1}{\exp\frac{h\nu}{kT_+} - 1}; \quad J_\nu^- = \frac{2\pi\nu^3}{c^2}\frac{1}{\exp\frac{h\nu}{kT_-} - 1}; \quad U = U^+ + U^-;$$

$$J^\pm = \int_0^\infty J_\nu^\pm d\nu = \frac{c}{2\pi}U^\pm; \quad U^\pm = \frac{2\sigma}{c}T_\pm^4; \quad J^\pm = J_{eq}\beta_\pm^{-4}, \tag{22}$$

i.e., the degree of nonequilibrium of the radiation will be characterized by the temperatures T_+ and T_- of quanta traveling in the positive and negative x directions, respectively. The absorption coefficient is again taken in a form [2] that corresponds to photo and bremsstrahlung absorption in the region of multiple and total ionization. Integrating (17) and (18) on the left and the right over ν from 0 to ∞ using (22), we have for the intensity J^+

$$\frac{1}{c}\frac{\partial J^\pm}{\partial t} \pm \frac{1}{2}\frac{\partial J^\pm}{\partial x} = \kappa_{eq}J_{eq} - \kappa_\pm J^\pm \tag{23}$$

or for the radiation densities

$$\frac{\partial U^\pm}{\partial t} \pm \frac{c}{2}\frac{\partial U^\pm}{\partial x} = c\left(\frac{1}{2}\kappa_{eq}U_{eq} - \kappa_\pm U^\pm\right). \tag{24}$$

Here κ_{eq} and $\kappa_\pm$ are given by the expressions

$$\kappa_{eq} = \frac{\int_0^\infty \kappa_\nu' U_{\nu eq}\, d\nu}{U_{eq}}, \quad \kappa_\pm = \frac{\int_0^\infty \kappa_\nu' U_\nu^\pm\, d\nu}{U^\pm}. \tag{25}$$

The total radiation density U and the flux S integrated over the spectrum are

$$U = U^+ + U^-, \quad S = \frac{c}{2}(U^+ - U^-). \tag{26}$$

The equations for finding them are obtained as usual by addition and subtraction of Eqs. (24):

$$\frac{\partial U}{\partial t} + \frac{\partial S}{\partial x} = c[\kappa_{eq}U_{eq} - (\kappa_+ U^+ + \kappa_- U^-)], \quad c\frac{\partial U}{\partial x} = -c[\kappa_+ U^+ - \kappa_- U^-]. \tag{27}$$

As follows from (25) κ_{eq} coincides completely with the definition of κ_{eq} given by formulas (9) and (11) in the diffusion approximation. The coefficients $\kappa_\pm$ are calculated from (10), where $\beta = \beta_\pm$ and $\beta_\pm = T/T_\pm$ should be set. We only need to note that according to the meaning itself of introducing the double-flow approximation the quantities $\beta_\pm$ can differ significantly from unity now, and therefore there can be no further simplification of the formulas for $\kappa_\pm$ analogous to passage from (10) to (12).

Let us now compare the frequency-averaged equations of the double-flow (27) and diffusion (7) approximations. A significant difference between them is that in (27) the terms $\kappa_+ U^+ \pm \kappa_- U^-$, upon replacing

$U^{\pm}$ in them using (26), contain not only U but also S while in (7) S is not involved in the corresponding sides of the equations. Because of this it is purposeless to change from (24) to (27) for the double-flow approximation; in solving various specific problems, we can use system (24) and prescribe its relationship with the remaining equations of gasdynamics in terms of (26). We also note that it is precisely in the irreducibility of (27) to a form analogous to (7) for $\beta_+ \neq \beta_-$ that the possibility of considering problems that are strongly anisotropic in radiation, for example, the problem of the cooling of a heated medium through a surface bordering a vacuum by luminescence, which is built into the double-flow approximation, manifests itself. On the other hand, when the anisotropy is weak (when $\beta_+ \sim \beta_-$) the double-flow approximation, as is well known, is practically no different from the diffusion approximation. All the above together provides a serious basis for using the double-flow approximation (needless to say, averaged in the manner presented above) in problems for which introduction of nonequilibrium transfer of radiation is dictated by necessity. Finally, we note that if the angular dependence of the radiation intensity is allowed for more accurately, generalization to the case of a large number of angles and, accordingly, radiation temperatures is evident.

NOTATION

$\bar{S}_\nu$, spectral radiation flux; $U_{\nu\text{eq}}$ and U_ν, equilibrium and nonequilibrium spectral radiation densities; c, velocity of light; κ'_ν, coefficient of absorption at the frequency ν corrected for forced emission; l'_ν, corresponding radiation mean free path; T, temperature of the substance; T_1, temperature of the radiation; h, Planck constant; σ, Stefan−Boltzmann constant; κ_0, absorption coefficient in the average-charge approximation; k, Boltzmann constant; J_m, ionization potential of a particle of multiplicity m; J, radiation intensity; $J_{\nu\text{eq}}$, equilibrium radiation intensity; Ω, solid angle; θ, angle between the direction of radiation propagation and the x axis; Ψ, step function determined by formula (1) of [2].

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